representation of a curve. Secondly, we expand it using Fourier theory. We can obtain alternative flavours by combining different curve representations and different Fourier expansions. Here, we shall consider Fourier descriptors of angular and complex contour representations. However, Fourier expansions can be developed for other curve representations (van Otterloo, 1991).

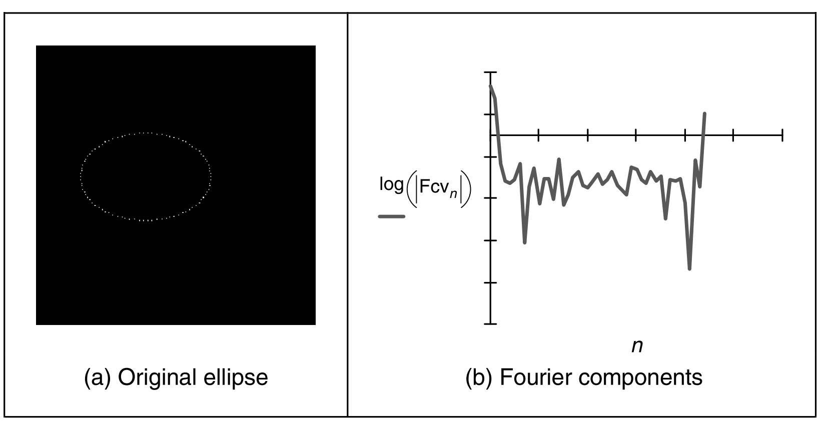
In addition to the curve's definition, a factor that influences the development and properties of the description is the choice of Fourier expansion. If we consider that the trace of a curve defines a periodic function, we can opt to use a Fourier series expansion. However, we could also consider that the description is not periodic. Thus, we could develop a representation based on the Fourier transform. In this case, we could use alternative Fourier integral definitions. Here, we will develop the presentation based on expansion in Fourier series. This is the common way used to describe shapes in pattern recognition.

It is important to notice that although a curve in an image is composed of discrete pixels, Fourier descriptors are developed for continuous curves. This is convenient since it leads to a discrete set of Fourier descriptors. We should also remember that the pixels in the image are the sampled points of a continuous curve in the scene. However, the formulation leads to the definition of the integral of a continuous curve. In practice, we do not have a continuous curve, but a sampled version. Thus, the expansion is approximated by means of numerical integration.

### 7.2.3.1 Basis of Fourier descriptors

In the most basic form, the coordinates of boundary pixels are $x$ and $y$ point coordinates. A Fourier description of these essentially gives the set of spatial frequencies that fit the boundary points. The *first* element of the Fourier components (the d.c. component) is simply the average value of the $x$ and $y$ coordinates, giving the coordinates of the centre point of the boundary, expressed in complex form. The *second* component essentially gives the radius of the circle that best fits the points. Accordingly, a circle can be described by its zero-order and first order components (the d.c. component and first harmonic). The *higher* order components increasingly describe detail, as they are associated with higher frequencies.

This is illustrated in Figure 7.6. Here, the Fourier description of the ellipse in Figure 7.6(a) is the frequency components in Figure 7.6(b), depicted in logarithmic form for purposes of display. The Fourier description has been obtained by using the coordinates of the ellipse boundary points. Here we can see that the low-order components dominate the description, as to be




(a) Original ellipse (b) Fourier components

**Figure 7.6** An ellipse and its Fourier description expected for such a smooth shape. In this way, we can derive a set a numbers that can be used to *recognize* the boundary of a shape: a similar ellipse should give a similar set of numbers, whereas a completely different shape will result in a completely different set of numbers.

We do, however, need to check the result. One way is to take the descriptors of a circle, since the first harmonic should be the circle's radius. A better way is to *reconstruct* the shape from its descriptors; if the reconstruction matches the original shape then the description would appear correct. We can reconstruct a shape from this Fourier description since the descriptors are *regenerative*. The zero-order component gives the position (or origin) of a shape. The ellipse can be reconstructed by adding in all spatial components, to extend and compact the shape along the *x*- and *y*-axes, respectively. By this inversion, we return to the original ellipse. When we include the zero and first descriptor, then we reconstruct a circle, as expected, shown in Figure 7.7(b). When we include all Fourier descriptors the reconstruction, Figure 7.7(c) is very close to the original Figure 7.7(a) with slight differences due to discretization effects.

(a) Original ellipse

(b) Reconstruction by zero and first order components

(c) Reconstruction by all Fourier components

**Figure 7.7** Reconstructing an ellipse from a Fourier description

This is only an outline of the basis to Fourier descriptors, since we have yet to consider descriptors that give the same description whatever an object's position, scale and rotation. Here we have just considered an object's description that is achieved in a manner that allows for reconstruction. To develop practically useful descriptors, we need to consider more basic properties. As such, we first turn to the use of Fourier theory for shape description.

#### 7.2.3.2 Fourier expansion

To define a Fourier expansion, we can start by considering that a continuous curve $c(t)$ can be expressed as a summation of the form

$$c(t) = \sum_k c_k f_k(t) \tag{7.1}$$

where $c_k$ defines the coefficients of the expansion, and the collection of functions $f_k(t)$ defines the basis functions. The expansion problem centres on finding the coefficients given a set of basis functions. This equation is very general and different basis functions can also be used. For example, $f_k(t)$ can be chosen such that the expansion defines a polynomial. Other bases define splines, Lagrange and Newton interpolant functions. A Fourier expansion represents periodic functions by a basis defined as a set of infinite complex exponentials. That is,

$$c(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \tag{7.2}$$

Here, $\omega$ defines the fundamental frequency and it is equal to $T/2\pi$, where $T$ is the period of the function. The main feature of the Fourier expansion is that it defines an orthogonal basis. This simply means that

$$\int_0^T f_k(t) f_j(t) \mathrm{d}t = 0 \tag{7.3}$$

for $k \neq j$. This property is important for two main reasons. First, it ensures that the expansion does not contain redundant information (each coefficient is *unique* and contains no information about the other components). Secondly, it simplifies the computation of the coefficients. That is, to solve for $c_k$ in Equation 7.1, we can simply multiply both sides by $f_k(t)$ and perform integration. Thus, the coefficients are given by

$$c_k = \int_0^T c(t) f_k(t) \Bigg/ \int_0^T f_k^2(t) \tag{7.4}$$

By considering the definition in Equation 7.2 we have:

$$c_k = \frac{1}{T} \int_0^T c(t) e^{-jk\omega t} \tag{7.5}$$

In addition to the exponential form given in Equation 7.2, the Fourier expansion can be expressed in trigonometric form. This form shows that the Fourier expansion corresponds to the summation of trigonometric functions that increase in frequency. It can be obtained by considering that

$$c(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k e^{jk\omega t} + c_{-k} e^{-jk\omega t} \right) \tag{7.6}$$

In this equation the values of $e^{jk\omega t}$ and $e^{-jk\omega t}$ define a pairs of complex conjugate vectors. Thus, $c_k$ and $c_{-k}$ describe a complex number and its conjugate. Let us define these numbers as

$$c_k = c_{k,1} - jc_{k,2} \quad \text{and} \quad c_{-k} = c_{k,1} + jc_{k,2} \tag{7.7}$$

By substitution of this definition in Equation 7.6 we obtain

$$c(t) = c_0 + 2 \sum_{k=1}^{\infty} \left( c_{k,1} \left( \frac{e^{jk\omega t} + e^{-jk\omega t}}{2} \right) + jc_{k,2} \left( \frac{-e^{jk\omega t} + e^{-jk\omega t}}{2} \right) \right) \tag{7.8}$$

That is,

$$c(t) = c_0 + 2 \sum_{k=1}^{\infty} (c_{k,1} \cos(k\omega t) + c_{k,2} \sin(k\omega t)) \tag{7.9}$$

If we define

$$a_k = 2c_{k,1} \quad \text{and} \quad b_k = 2c_{k,2} \tag{7.10}$$

we obtain the standard trigonometric form given by

$$c(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\omega t) + b_k \sin(k\omega t)) \tag{7.11}$$

The coefficients of this expansion, $a_k$ and $b_k$, are known as the *Fourier descriptors*. These control the amount of each frequency that contributes to make up the curve. Accordingly, these descriptors can be said to *describe* the curve, since they do not have the same values for different curves. Notice that according to Equations 7.7 and 7.10 the coefficients of the trigonometric and exponential form are related by

$$c_k = \frac{a_k - jb_k}{2} \quad \text{and} \quad c_{-k} = \frac{a_k + jb_k}{2} \tag{7.12}$$

The coefficients in Equation 7.11 can be obtained by considering the orthogonal property in Equation 7.3. Thus, one way to compute values for the descriptors is

$$a_k = \frac{2}{T}\int_0^T c(t)\cos(k\omega t)\,\mathrm{d}t \quad \text{and} \quad b_k = \frac{2}{T}\int_0^T c(t)\sin(k\omega t)\,\mathrm{d}t \tag{7.13}$$

To obtain the Fourier descriptors, a curve can be represented by the complex exponential form of Equation 7.2 or by the sin/cos relationship of Equation 7.11. The descriptors obtained by using either of the two definitions are equivalent, and they can be related by the definitions of Equation 7.12. In general, Equation 7.13 is used to compute the coefficients since it has a more intuitive form. However, some works have considered the complex form (e.g. Granlund, 1972). The complex form provides an elegant development of rotation analysis.

#### *7.2.3.3 Shift invariance*

Chain codes required special attention to give start point invariance. Let us see whether that is required here. The main question is whether the descriptors will change when the curve is shifted. In addition to Equations 7.2 and 7.11, a Fourier expansion can be written in another sinusoidal form. If we consider that

$$|c_k| = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \varphi_k = a\tan^{-1}(b_k/a_k) \tag{7.14}$$

then the Fourier expansion can be written as

$$c(t) = \frac{a_0}{2} + \sum_{k=0}^{\infty} |c_k|\cos(k\omega t + \varphi_k) \tag{7.15}$$

Here, $|c_k|$ is the *amplitude* and $\varphi_k$ is the *phase* of the Fourier coefficient. An important property of the Fourier expansion is that $|c_k|$ does not change when the function $c(t)$ is shifted (i.e. translated), as in Section 2.6.1. This can be observed by considering the definition of Equation 7.13 for a shifted curve $c(t+\alpha)$. Here, $\alpha$ represents the shift value. Thus,

$$a'_k = \frac{2}{T}\int_0^T c(t'+\alpha)\cos(k\omega t')\,\mathrm{d}t \quad \text{and} \quad b'_k = \frac{2}{T}\int_0^T c(t'+\alpha)\sin(k\omega t')\,\mathrm{d}t \tag{7.16}$$

By defining a change of variable by $t = t' + \alpha$, we have

$$a'_k = \frac{2}{T}\int_0^T c(t)\cos(k\omega t - k\omega\alpha)\,\mathrm{d}t \quad \text{and} \quad b'_k = \frac{2}{T}\int_0^T c(t)\sin(k\omega t - k\omega\alpha)\,\mathrm{d}t \tag{7.17}$$

After some algebraic manipulation we obtain

$$a'_k = a_k\cos(k\omega\alpha) + b_k\sin(k\omega\alpha) \quad \text{and} \quad b'_k = b_k\cos(k\omega\alpha) - a_k\sin(k\omega\alpha) \tag{7.18}$$

The amplitude $|c'_k|$ is given by

$$|c'_k| = \sqrt{(a_k \cos(k\omega\alpha) + b_k \sin(k\omega\alpha))^2 + (b_k \cos(k\omega\alpha) - a_k \sin(k\omega\alpha))^2} \tag{7.19}$$

That is,

$$|c'_k| = \sqrt{a_k^2 + b_k^2} \tag{7.20}$$

Thus, the amplitude is independent of the shift $\alpha$. Although shift invariance could be incorrectly related to translation invariance, as we shall see, this property is related to rotation invariance in shape description.

#### *7.2.3.4 Discrete computation*

Before defining Fourier descriptors, we must consider the numerical procedure necessary to obtain the Fourier coefficients of a curve. The problem is that Equations 7.11 and 7.13 are defined for a *continuous* curve. However, given the discrete nature of the image, the curve $c(t)$ will be described by a collection of *points*. This discretization has two important effects. First, it limits the number of frequencies in the expansion. Secondly, it forces numerical approximation to the integral defining the coefficients.

Figure 7.8 shows an example of a discrete approximation of a curve. Figure 7.8(a) shows a continuous curve in a period, or interval, $T$. Figure 7.8(b) shows the approximation of the curve by a set of discrete points. If we try to obtain the curve from the sampled points, we will find that the sampling process reduces the amount of detail. According to the Nyquist theorem, the maximum frequency $f_c$ in a function is related to the sample period $\tau$ by

$$\tau = \frac{1}{2f_c} \tag{7.21}$$

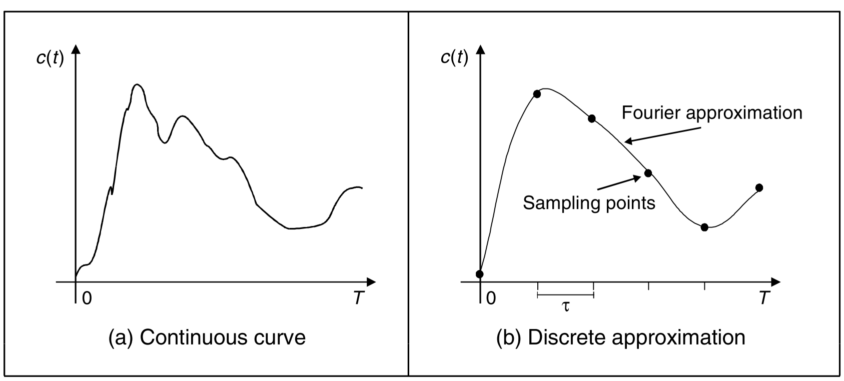



**Figure 7.8** Example of a discrete approximation

Thus, if we have $m$ sampling points, then the sampling period is equal to $\tau = T/m$. Accordingly, the maximum frequency in the approximation is given by

$$f_c = \frac{m}{2T} \tag{7.22}$$